APPROXIMATE DESCRIPTION OF THE VIBRATION OF RODS OF PERIODIC STRUCTURE WITH MASS OSCILLATORS

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A great deal of research has been devoted to the problem of the construction of approximate equations for the description of static dynamic problems of the mechanics of a continuous medium with a periodic structure by making use of averaging. The general principle for the construction of such approximations and the confirmation of their convergence has been formulated in [1-7]. The initial problem contains a small parameter characterizing the period size. The essence of the method lies in the fact that the desired solution is approximated in the form of the sum of continuous and rapidly oscillating components. A method is proposed in this paper for the construction of approximate equations for a system which describes the vibrations of a rod of periodic structure with continuously distributed mass oscillators [8]. The system of equations can simulate the longitudinal motion of a rod structural element with masses supporting a functional load attached to it. The problem of the closeness of the approximate solutions to the exact one is investigated. An estimate of the convergence is obtained.

Let us consider the problem of longitudinal vibrations of a rod of periodic structure with continuously distributed mass oscillators

$$u_{\varepsilon tt}-\frac{\partial}{\partial x}\left(a\left(\frac{x}{\varepsilon}\right)u_{\varepsilon x}\right)=-\int\limits_0^\infty m(\omega)\,v_{\varepsilon tt}d\omega+f \text{ in } Q,$$
$$v_{\varepsilon tt}+\omega^2(v_\varepsilon-u_\varepsilon)=0 \text{ in } \{m(\omega)>0\}\times Q, \tag{1}$$

$u_\varepsilon = u_{\varepsilon t} = v_\varepsilon = v_{\varepsilon t} = 0$ at $t = 0$ and $u_\varepsilon = 0$ at $x = 0$ and l. Here $u_\varepsilon(t, x)$ and $v_\varepsilon(t, x, \omega)$ are the displacements of the elastic rod and mass oscillators, respectively, $m(\omega) \geqslant 0$ is the density of the distribution of mass oscillators, ω is a parameter (eigenfrequency), ε is a small parameter which characterizes the periodicity of the elastic properties of the rod, $a(s) \geqslant a_0 > 0$ is a periodic function with the period unity, a_0 is a constant, $f(t, x)$ is the external load, and $Q = (0, l) \times (0, T)$. It is assumed that $a \in L^\infty(E)$, and E is the period.

If $m \equiv 0$, then the second equation of the system does not have to be considered, and all the following discussions will pertain to the mixed problem for the one-dimensional wave equation.

Let us denote $y = x/\varepsilon$ and consider the problem in $\varphi_1(y)$ which is periodic in y,

$$-\frac{\partial}{\partial y}\left(a(y)\frac{\partial}{\partial y}(\varphi_1-y)\right)=0,\quad \langle\varphi_1\rangle=0, \tag{2}$$

where $\langle\cdot\rangle$ is the average value of a function with the period E. Let us also set $q = \langle a(1-\varphi_{1y})\rangle$, and let u and v be the solution of the following problems:

$$u_{tt}-qu_{xx}=-\int\limits_0^\infty m(\omega)\,v_{tt}d\omega+f \text{ in } Q,$$
$$v_{tt}+\omega^2v=\omega^2u \text{ in } \{m(\omega)>0\}\times Q, \tag{3}$$
$$u=u_t=v=v_t=0 \text{ for } t=0;\ u=0 \text{ for } x=0,\ l.$$

The main result of this paper is a proof of the confirmation of the fact that the solution of the problem (1) converges under the appropriate conditions to the solution of the problem

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(3) as $\varepsilon \to 0$. The nature of the convergence is determined below, and exact statements are formulated. First, the existence of a solution of the problem (1) will be proven, and then the question of the construction of approximate solutions and their convergence will be discussed.

Let $H^k(\Omega)$ be the Sobolev space of the functions which are integrable in Ω with second order along with their own derivatives out to order k, $\Omega = (0, l)$, and $\overset{\circ}{H}{}^1_0(\Omega) = \{\varphi \in H^1(\Omega) | \varphi = 0,\ x = 0,\ l\}$. We will denote as B_i the function space with norm $\|\psi\|^2_{B_i} = \int_0^\infty m(\omega)\omega^i \|\psi(\omega)\|_0^2 d\omega$, $i = 0, 2$, where $\|\cdot\|_k$ is the norm in $H^k(\Omega)$. In the following the conditions on m will guarantee nonemptiness of the space B_i. Now let us return to the problem (1) and obtain *a priori* estimates of the solution. Let us multiply the second equation of the system (1) by v'_ε in scalar fashion in $L^2(\Omega)$ (we denote a derivative with respect to t with a prime)

$$\frac{d}{dt}\left(\|v'_\varepsilon(t)\|_0^2 + \omega^2\|v_\varepsilon(t)\|_0^2\right) = 2\omega^2\left(u_\varepsilon(t), v'_\varepsilon(t)\right).$$

Taking the relationship $(u_\varepsilon(\tau), v_\varepsilon(\tau))_\tau = (u'_\varepsilon(\tau), v_\varepsilon(\tau)) + (u_\varepsilon(\tau), v'_\varepsilon(\tau))$ into account and using the inequalities of Cauchy and Young, we thence derive

$$\|v'_\varepsilon(t)\|_0^2 + \omega^2\|v_\varepsilon(t)\|_0^2 \leqslant c_0\omega^2\left\{\|u_\varepsilon(t)\|_0^2 + \int_0^t \left(\|u'_\varepsilon(\tau)\|_0^2 + \|u_\varepsilon(\tau)\|_0^2\right) d\tau\right\}, \tag{4}$$

where the constant c_0 depends only on T. The dependence of the function v_ε on the parameter ω will not be shown here. Now we note that when the second equation of the system (1) is taken into account one can write the first one in the form

$$L_\varepsilon u_\varepsilon \equiv u''_\varepsilon - \frac{\partial}{\partial x}\left(a\left(\frac{x}{\varepsilon}\right)u_{\varepsilon x}\right) + \alpha u_\varepsilon = \int_0^\infty m\omega^2 v_\varepsilon d\omega + f,\ \alpha = \int_0^\infty m\omega^2 d\omega. \tag{5}$$

We will assume for the present that all the norms which arise are finite and integrals over ω are convergent. Let us multiply (5) by u'_ε, evaluate the terms on the right-hand side according to the Cauchy inequality, and make use of (4). Then we will arrive at an inequality of the Gronwall type. Integration of it yields

$$\max_{0 \leqslant t \leqslant T}\{\|u'_\varepsilon(t)\|_0^2 + \|u_\varepsilon(t)\|_1^2\} \leqslant c_1, \tag{6}$$

and c_1 depends on m, a_0, T, and f. If one now multiplies (4) by m and integrates over ω from 0 to ∞, then we will have

$$\max_{0 \leqslant t \leqslant T}\{\|v'_\varepsilon(t)\|_{B_0}^2 + \|v_\varepsilon(t)\|_{B_2}^2\} \leqslant c_2. \tag{7}$$

<u>LEMMA 1.</u> Let $m(\omega)$, $m(\omega)\omega^2 \in L^1(0, \infty)$; $f \in L^2(Q)$. Then there exists a unique solution of the problem (1), and

$$u_\varepsilon \in L^\infty\left(0, T;\ \overset{\circ}{H}{}^1_0(\Omega)\right),\ u_{\varepsilon t} \in L^\infty(0,\ T;\ L^2(\Omega)),\ v_\varepsilon \in L^\infty(0,\ T;\ B_2),\ v_{\varepsilon t} \in$$
$$\in L^\infty(0, T; B_0).$$

Let us choose the basis $\{\psi_j\}$ ($j = 1, 2, 3, \ldots$) in the space $\overset{\circ}{H}{}^1_0(\Omega)$ for the existence proof, and we will seek Galerkin approximations in the form

$$u_{\varepsilon n}(t) = \sum_{i=1}^n q_{in}(t)\psi_i,\ v_{\varepsilon n}(t) = \sum_{i=1}^n p_{in}(t, \omega)\psi_i.$$

The functions q_{in} and p_{in} satisfy the system of ordinary differential equations

$$\left(u''_{\varepsilon n}, \psi_j\right) + \left(a\left(\frac{x}{\varepsilon}\right)u_{\varepsilon nx}, \psi_{jx}\right) + \alpha(u_{\varepsilon n}, \psi_j) = \int_0^\infty m\omega^2(v_{\varepsilon n}, \psi_j) d\omega + (f, \psi_j),$$
$$\left(v''_{\varepsilon n} + \omega^2(v_{\varepsilon n} - u_{\varepsilon n}), \psi_j\right) = 0,\ 1 \leqslant j \leqslant n,$$

$$u_{\varepsilon n}(0) = u'_{\varepsilon n}(0) = v_{\varepsilon n}(0) = v'_{\varepsilon n}(0) = 0.$$

Here $(.,.)$ denotes the scalar product in $L^2(\Omega)$. The estimates (6) and (7) are valid for $u_{\varepsilon n}$ and $v_{\varepsilon n}$, since one can multiply the second Eq. (1) and (5) by v'_ε and u'_ε, respectively, for the approximate solutions. One can choose the constants c_1 and c_2 to be independent of n. This will guarantee the solvability of the approximate equations on [0, T]. The question of the construction of Galerkin approximations in a similar situation has been discussed in detail in [9]. According to the estimates (6) and (7), there exists a sequence $u_{\varepsilon s}$, $v_{\varepsilon s}$ such that as $s \to \infty$ $u_{\varepsilon s} \to u_\varepsilon$, $u'_{\varepsilon s} \to u'_\varepsilon$, $v_{\varepsilon s} \to v_\varepsilon$, $v'_{\varepsilon s} \to v'_\varepsilon$ *-weakly in $L^\infty(0, T; H_0^1(\Omega))$, $L^\infty(0, T; L^2(\Omega))$, $L^\infty(0, T; B_0)$, respectively. The limiting functions u_ε and v_ε will satisfy (5) and the second equation of the system (1) in the sense of integral identities.

By virtue of the linearity of the problem, the uniqueness proof is carried out as usual for the difference of two possible solutions.

In order to obtain higher differentiability of the solution, it is necessary to impose additional conditions on the function $a(s)$. As a rule, $a(s)$ is a discontinuous function in applications, and therefore the requirement $a \in L^\infty(E)$ is a natural one. In particular, it follows from the estimates obtained that the stresses in the rod $\sigma_\varepsilon = a u_{\varepsilon x}$ are bounded in $L^2(Q)$ (a is Young's modulus). Concerning the accelerations $u_{\varepsilon tt}$, they are bounded only in the space $L^\infty(0, T; H^{-1}(\Omega))$, $H^{-1}(\Omega)$, is conjugate to $H_0^1(\Omega)$. However, for the problem (3), where $q > 0$ is a constant, the estimate u_{tt} occurs in more differentiable classes. Notably, we set $\|\psi\|_{F_i}^2 = \int_0^\infty m\omega^i \|\psi(\omega)\|_1^2 d\omega$, $i = 0, 2$.

LEMMA 2. Let $f \in L^2(0, T; H_0^1(\Omega))$ under the conditions of Lemma 1. Then the inclusions

$$u \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega)),\ u_t \in L^\infty(0, T; H_0^1(\Omega)),$$
$$v \in L^\infty(0, T; F_2),\ v_t \in L^\infty(0, T; F_0)$$

occur for the solution of the problem (3).

Proof. Let us consider the equation for u and v analogous to (5) and multiply it by u'_{xx}. Let us also multiply the second equation of the system (3) by v'_{xx}. Taking the equation $(f(t), u'_{xx}(t)) = -(f_x(t), u'_x(t))$ into account, we note that these two relationships are completely analogous to those which were used in the derivation of (6) and (7) and can be formally derived from them. To do this, one should replace the norm in $L^2(\Omega)$ everywhere by the norm in $H_0^1(\Omega)$ and so forth. The difference consists of the fact that the differentiability in x is now greater by one. We thereby obtain estimates similar to (6) and (7):

$$\max_{0 \le t \le T} \{\|u'_x(t)\|_0^2 + \|u_{xx}(t)\|_0^2\} \le \bar{c}_1, \quad \max_{0 \le t \le T} \{\|v'_x(t)\|_{B_0}^2 + \|v_x(t)\|_{B_2}^2\} \le \bar{c}_2.$$

The constants $\bar{c}_i$ depend on m, q, T, and f. From this $u_{xx} \in L^2(Q)$ follows, and therefore we have the inclusion $u_{tt} \in L^2(Q)$ from an equation of the form (5).

One can study the question of a further increase in the differentiability of the solution in a similar fashion.

Now let us construct an approximate solution of the problem (1) and prove the assertion about its convergence. We will set

$$\bar{u}_\varepsilon = u + \varepsilon w_1 + \varepsilon^2 w_2,$$
$$u = u(t, x),\ w_1 = u - \varphi_1(y) u_x,\ w_2 = u - \varphi_1(y) u_x + \varphi_2(y) u_{xx}. \tag{8}$$

A function φ_2 which is periodic in y is determined as the solution of the problem $(a\varphi_{2y})_y = (a\varphi_1)_y$, $\langle \varphi_2 \rangle = 0$, which has, just as does (2), a unique solution in the space $H^1(E)$ of functions which are periodic in y; $\max_E |\varphi_i| \le c\|\varphi_i\|_1$ $(i = 1, 2)$, and c does not depend on ε. Now we note the equation

$$-\frac{\partial}{\partial x}\left(a\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x}\right) = \varepsilon^{-2}A_2 + \varepsilon^{-1}A_1 + \varepsilon^0 A_0,$$

$$A_2 = -\frac{\partial}{\partial y}\left(a(y)\frac{\partial}{\partial y}\right),\quad A_1 = -\frac{\partial}{\partial x}\left(a(y)\frac{\partial}{\partial y}\right) - \frac{\partial}{\partial y}\left(a(y)\frac{\partial}{\partial x}\right),\quad A_0 = -a(y)\frac{\partial^2}{\partial x^2}.$$

Taking this expansion into account, let us substitute $\bar{u}_\varepsilon$ into (5) and determine u and v from the equations

$$L_\varepsilon \bar{u}_\varepsilon = f + \int_0^\infty m\omega^2 v\,d\omega + \varepsilon h_1 + \varepsilon^2 h_2,\quad v_{tt} + \omega^2(v-u) = 0$$

with null initial and boundary conditions. Here

$$h_1 = \alpha w_1 + w_{1tt} + A_1 w_2 + A_0 w_1,\quad h_2 = \alpha w_2 + w_{2tt} + A_0 w_2.$$

One should consider the variables x and y in these equations to be independent. Then we obtain

$$L_\varepsilon \bar{e}_\varepsilon = \int_0^\infty m\omega^2 d_\varepsilon\,d\omega - \varepsilon h_1 - \varepsilon^2 h_2,$$

$$d_{\varepsilon tt} + \omega^2(d_\varepsilon - \bar{e}_\varepsilon - \varepsilon\lambda_\varepsilon) = 0,\quad \lambda_\varepsilon = w_1 + \varepsilon w_2$$

for $\bar{e}_\varepsilon = u_\varepsilon - \bar{u}_\varepsilon$ and $d_\varepsilon = v_\varepsilon - v$. We note in this connection that the initial data for $\bar{e}_\varepsilon$ and $\varepsilon\lambda_\varepsilon$ are null, and the boundary conditions agree.

THEOREM. Let the conditions of Lemma 1 be satisfied, and in addition let $u \in H^5(Q)$, and $m\omega^4 \in L^1(0, \infty)$. Then we obtain

$$\max_{0\leqslant t\leqslant T}\{\|u_\varepsilon(t) - \bar{u}_\varepsilon(t)\|_1 + \|u_{\varepsilon t}(t) - u_t(t)\|_0\} \leqslant c_3\varepsilon^{1/2},$$
$$\max_{0\leqslant t\leqslant T}\{\|v_\varepsilon(t) - v(t)\|_{B_2} + \|v_{\varepsilon t}(t) - v_t(t)\|_{B_0}\} \leqslant c_4\varepsilon^{1/2}. \tag{9}$$

The constants c_3 and c_4 do not depend upon ε.

PROOF. Let us denote as $\bar{\Lambda}_\varepsilon$ the continuation of the function λ_ε from the boundary $\{t=0\} \cup \{x = 0, l\}$ into the region Q such that $\|\bar{\Lambda}_\varepsilon\|_3 \leqslant c_5$, and c_5 does not depend upon ε for all $\varepsilon \leqslant \varepsilon_0$. This is possible by virtue of the indicated estimate on $|\varphi_i|$ and the existing differentiability of the function u. Let us set $\Lambda_\varepsilon = \lambda_\varepsilon - \bar{\Lambda}_\varepsilon$. Then for $e_\varepsilon = \bar{e}_\varepsilon + \varepsilon\bar{\Lambda}_\varepsilon$ and d_ε we obtain a problem with the homogeneous conditions

$$L_\varepsilon e_\varepsilon = \int_0^\infty m\omega^2 d_\varepsilon\,d\omega - \varepsilon h_1 - \varepsilon^2 h_2 + \varepsilon L_\varepsilon \bar{\Lambda}_\varepsilon, \tag{10}$$
$$d_{\varepsilon tt} + \omega^2(d_\varepsilon - e_\varepsilon - \varepsilon\Lambda_\varepsilon) = 0,$$

$e_\varepsilon = e_{\varepsilon t} = d_\varepsilon = d_{\varepsilon t} = 0$ at $t = 0$, and $e_\varepsilon = 0$ at $x = 0, l$. Since $a \in L^\infty(E)$, the right-hand part of the first equation here along with its own first derivative with respect to t belongs to the space $L^\infty(0, T; H^{-1}(\Omega))$. Consequently, for $t \in [0, T]$ we have the estimate [10]

$$\|e_{\varepsilon t}(t)\|_{-1}^2 + \|e_\varepsilon(t)\|_0^2 \leqslant c_6\varepsilon\int_0^t \|h_1 + \varepsilon h_2 - L_\varepsilon\bar{\Lambda}_\varepsilon\|_{-1}^2 d\tau + c_7\varepsilon\int_0^t\int_0^\infty m\omega^2\|d_\varepsilon\|_{-1}^2 d\omega\,d\tau. \tag{11}$$

However, similarly to (4) one can obtain from the first equation of the system (10)

$$\|d_{\varepsilon t}(t)\|_{-1}^2 + \omega^2\|d_\varepsilon(t)\|_{-1}^2 \leqslant c_0\omega^2\left(\|e_\varepsilon(t) + \varepsilon\Lambda_\varepsilon(t)\|_{-1}^2 + \int_0^t\left(\|e_\varepsilon(\tau) + \varepsilon\Lambda_\varepsilon(\tau)\|_{-1}^2 + \|e_{\varepsilon\tau}(\tau) + \varepsilon\Lambda_{\varepsilon\tau}(\tau)\|_{-1}^2\right)d\tau\right). \tag{12}$$

Therefore we will have from this and from (11)

$$\max_{0\leq t\leq T}\{\|e_{\varepsilon t}(t)\|_{-1}^2+\|e_\varepsilon(t)\|_0^2\}\leqslant c_8\varepsilon$$

by virtue of the Gronwall lemma. Similarly, differentiating the first equation of the system (10) with respect to t, one can show

$$\max_{0\leq t\leq T}\{\|e_{\varepsilon tt}(t)\|_{-1}^2+\|e_{\varepsilon t}(t)\|_0^2\}\leqslant c_9\varepsilon. \tag{13}$$

At the same time after multiplication by m and integration over ω from 0 to ∞ we obtain from (12)

$$\max_{0\leq t\leq T}\|d_\varepsilon(t)\|_{E_2}^2\leqslant c_{10}\varepsilon,\ \ \|\psi\|_{E_2}^2=\int\limits_0^\infty m\omega^2\|\psi(\omega)\|_{-1}^2d\omega. \tag{14}$$

Since the operator $\frac{\partial}{\partial x}\left(a\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x}\right)$ produces a mutually single-valued and mutually continuous mapping between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ with the appropriate estimate, we obtain

$$\max_{0\leq t\leq T}\|e_\varepsilon(t)\|_1^2\leqslant c_{11}\varepsilon$$

from (13) and (14) and the first equation of the system (10). The second inequality (9) follows from this with (13) and a relationship for d_ε of the kind (4) taken into account.

The theorem is proved. Thus the convergence of the solution of the problem (1) in the form (9) to the solution of the problem (3) as $\varepsilon\to 0$ has been established. The possibility of approximating the solution of the problem of vibrations of a rod of periodic structure by the problem of vibrations of a uniform rod has thereby been proven. The physical parameter q of the latter is found with the help of the procedure indicated in the paper. We also note that an approximate analysis of some problems for Eqs. (3) has been performed in [8]. The solutions derived in this paper can be used in Eq. (8) to construct approximations $\bar{u}_\varepsilon$ of the problem (1).

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